Charles Tapiero and Pierre Vallois

# Fractional and Statistical Randomness

*Prisme* N°38
March 2019

The Cournot Centre and Foundation

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# Fractional and Statistical Randomness*

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* This joint paper by Pierre Vallois and Charles Tapiero is based on a series of joint papers and in particular: "Fractional Randomness" (*Physica A: Statistical Mechanics and its Applications*, 2016); "Implied Fractional Hazard Rates and Default Risk Distributions" (*Probability, Uncertainty and Quantitative Risk*, 2017); "Fractional Randomness and the Brownian Bridge" (*Physica A: Statistical Mechanics and its Applications*, 2018a), and finally "Randomness and Fractional Alpha-Stable Distributions" (*Physica A: Statistical Mechanics and its Applications*, 2018b).

# 1. Introduction

Financial practice is based on statistical data, digitalization, information technologies, data analytics, and financial and economic theories. Globalization, computers on steroids, data granularity and the rising complexity of financial systems in transition are challenging further the practice of finance. The intent of this text is two-fold: on the one hand, to introduce elements of fractional calculus applied to statistical distributions, and, on the other, to define a fractional randomness. Scaling defines a time measure with respect to which data are recorded, on the basis of which statistical analyses are made. For example, a model based on day data, defining a stock price from day-to-day, differs from one defined in an intraday, weekly or any other time frame. As a result, statistical implications from models' data depend on the time scale. For example, intraday data (measured by milliseconds, minutes and hours) exhibit a randomness that differs due to data's granularity. By the same token, trading strategies are exchange mechanisms that account for the time granularity. When time scales change, prices change. In such contexts, high-frequency and day-data trading differ fundamentally due to their time scales.

Consider the speed at which a train travels. A fast train (say the TGV in France) recording images as it travels at high speeds has little granular detail relative to a "slower" train. Yet, they both observe the same landscape, each providing a granular snapshot. Each sample time and its granularity, seeking to reconcile theoretical assumptions that sampled data and the information it implies provide. Fractional calculus is then a means to reconcile models' granularities defined by fractional operators and their "fractional index". For example, fractional and statistical operators are applied to probability distributions and stochastic processes that provide alternative definitions of "fractional randomness".

# 2. Origins of fractional calculus

The origins of fractional calculus, its applications and its extensions to stochastic systems are not new. Cauchy, for example, provided an expression for the remainder of a Taylor series integer residual, accounting for terms neglected by Taylor series' approximations. Leibniz (1695) extended Cauchy's remainder to a fractional remainder, thereby expanding the scope and the meaning of this remainder and providing an initial foundation for fractional calculus. The product rule:

$$d^k(xy) = d^k x d^0 y + \frac{k}{1!} d^{k-1} x d^1 y + \frac{k(k-1)}{2!} d^{k-2} x d^2 y + \cdots$$

led Leibniz to consider a fractional development, similar to that of a binomial expansion given below.[1]

$$(x+y)^k = x^k y^0 + \frac{k}{1!} x^{k-1} y + \frac{k(k-1)}{2!} x^{k-2} y^2 + \ldots$$

with $(x^0, y^0) = (1,1)$.

This leads further to the fractional binomial expansion,

$$(x + y^H = x^H \left(1 + \frac{y}{x}\right)^H = x^H \left(1 + \frac{H}{1!}\left(\frac{y}{x}\right) + \frac{H(H-1)}{2!}\left(\frac{y}{x}\right)^2 + \cdots\right)$$

$$= x^H + \frac{Hx^{H-1}}{1!} + \frac{H(H-1)}{2!} x^{H-2} y^2 + \cdots$$

Although Leibniz did not develop a fractional calculus, he raised modelling and computational issues for mathematicians to reckon with. A summarized historical review includes, but is not exclusive to the following list:

[1] See Hilfer (1995) (2000) for a review.

- Cauchy defined theTaylor series remainder;
- Leibniz (1695) investigated fractional derivatives;
- Euler (1738) introduced the gamma function and its integral;
- Fourier (1822) provided a trigonometric definition of a fractional derivative;
- Liouville (1832) defined fractional operators;
- Riemann provided a definition similar to that of Liouville that differs by limits of integration;
- Grunwald (1867) and Letnikov (1868) provided a difference approximation to the fractional derivative;
- Following Hadamard, Marchaud developed fractional differential equations (1927);
- Hurst published his empirical rescaled range methodology (Hurst index, defined using Nile data) (1951);
- Caputo (1967) provided a definition of fractional integrals (pointed out by Liouville);
- Mandelbrot and colleagues (1963), (1967), (1968) and Fama (1963) developed fractional Brownian motion.

Since then, fractional Poisson (Laskin, 2003), Duncan et al (2000), fractional Brownian Motion (Mandelbrot et al. (1963, 1967, 1968), fractional probability distributions and statistical fractional randomness defined by a Fractional Brownian Bridge (with fractional index $1/2 < H < 1$(with fractional index $0 < H < 1/2$ ) have been introduced by Tapiero and Vallois (2016) (2017) (2018a) (2018b). There is, in addition, a long series of applications.[2] and developments by Potlubny (1999), Caputo (1967), Metzler and Klafter (2004),

[2] Additional reviews and references include Hilfer (ed.) (2000) and Duncan, Hu and Pasik-Duncan (2000). See also, Bjork and Hult (2005), Cheridito (2001), Dacorogna et al. (1993), Muller (1992), Muller et al. (1990) (1993), and Dung (2008).

Additional references to fractional calculus abound, including, for example, contributions by Baleanu et al. (2010), Almeida, Pooseh and Torres (2015), and many others, broadly available in academic papers and books. Other discrete, continuous deterministic, as well as stochastic diffusion systems are defined by either differential equations, Fokker–Plank equations or other partial differential equations.[3]

There are numerous applications to fractional calculus and their fractional operators altering the traditional rules of the differential calculus. They introduce both an opportunity to expand these rules and, at the same time, challenge their interpretation (and thus usefulness). For example, an application of Liouville operators to conventional probability distributions need not necessarily lead to conventional probability distributions. However, a fractional hazard rate as well as Brownian Motion imply complete probability distributions (Tapiero and Vallois, 2017). Extensions and applications to such developments have been made by numerous authors, as stated above.[4]

# 3. Fractional Operators

Fractional calculus is based on integral operators $I_a^H$ and $D_a^H$. Considering the fractional derivative $0 < H < 1$ and $a \geq 0$, they are defined by the Riemann–Liouville functions:

[3] See, for example, Laskin (2003), Orsingher and Polito (2012), Miller and Ross (1993), Taqqu (1986) (2003), and Jumarie (2005a) (2005b) (2006a) (2006b) (2009) (2010) (2013).

[4] See Hurst (1951) for a fractional index based on R/S large samples analysis; see also Imhoff (1985), Tapiero and Vallois (1996) (1997), Vallois and Tapiero (1996), (2000), (2008). For ARFIMA regressions and other econometric models, see Baillie (1996), Baum et al. (1999), Beran (1992), Engle (1987), Engle, Itô and Lin (1990) and Bollerslev (1986). For applications applied to long-run dependence in stock prices, see Granger and Joyeux (1980) and Green and Felitz (1977).

$$I_a^H(t) = \frac{1}{\Gamma(H)} \int_a^t (t-\tau)^{H-1} f(\tau) d\tau, t \geq a$$

where $f: [a, \infty[ \rightarrow \mathfrak{R}$ is a function such that $\int_a^t |f(\tau)|\, d\tau < \infty$ for any $t \geq a$. With the fractional derivative for $f$ given by:

$D_a^H(f) = DI_a^{1-H} f$ for $0 < H \leq 1$ where $D$ is the usual derivative. When $a = 0$, we write $I_a^H = I_0^H = I^H$ and $D_a^H = D_0^H = D^H$, in which case,

$$D^H I^H(f) = f. \qquad I^H D^H(f). \qquad 0 < H \leq 1.$$

Further note that at $H = 1, I^1(f)(t) = \int_0^t f\,(\tau) d\tau$ and $D^1 = D$.

Consider, for example, the function $\varphi_\beta(t)$:

$$\varphi_\beta(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1}, x > 0, \beta > 0\,.$$

Then,

$$I^H \varphi_\beta(t) = \frac{1}{\Gamma(H)\Gamma(\beta)} \int_0^t (t-\tau)^{H-1}\, \tau^{\beta-1} d\tau\,.$$

We set $\tau = ut$

$$I^H \varphi_\beta(t) = \frac{t^{H+\beta-1}}{\Gamma(H)\Gamma(\beta)} \int_0^1 (1-u)^{H-1} u^{\beta-1} du = \frac{t^{H+\beta-1}}{\Gamma(H)\Gamma(\beta)} \varphi_{\beta+H}(t).$$

Therefore,

$$D^H \varphi_\beta = DI^{1-H} \varphi_\beta = D\varphi_{\beta+1-H} = \varphi_{\beta-H}\,.$$

When $f$ is a probability distribution, its fractional distribution may or may not exist (as we shall prove). We define a Fractional Cumulative Density Function (FCDF) and a Fractional Probability Distribution (FPD) as follows:

**Definition 1**. Let $0 < H \leq 1$.

Let $f(t)$ be a non-negative function and $F(t)$ a non-decreasing function such that $F(o) = 0$. The FCDF and the FPD associated with $f(t)$ are:

$$F_H(t) = I^H(f)(t) = \frac{1}{\Gamma(H)} \int_0^t (t-\tau)^{H-1} f(\tau) d\tau, t \geq 0$$

and

$$f_H(t) = D^H(F)(t) = I^{1-H} f(t) = \frac{1}{\Gamma(1-H)} \int_0^t (t-\tau)^{-H} f(\tau) d\tau, t \geq 0.$$

The proof for $f_H(t)$ above follows from the following elementary fractional operators:

$$f_H = D^H F = D^1 I^{1-H} I^1 f = D^1 I^1 I^{1-H} f = I^{1-H} f.$$

Note that, an FPD is defined as the fractional derivative of its Cumulative Distribution Function (CDF) while the FCDF is defined by the fractional integration of the probability distribution. These distributions, defined by their fractional operators, may then be calculated numerically by the computational approach used for a fractional derivative and its integration. Intuitively, we define as well:

$$\frac{dF_H}{dt^H} = D^H F(t) = f_H(t) \text{ and } F_H = \int_0^t f(t)(dt)^H$$

where time intervals of order $(dt)^H$ emphasize a difference to standard integral calculus. Thus, $H$ corresponds to a reference and standard definition of an integral, where $H > 1$ corresponds to an integral based on a "more refined" estimate of its CDF. Inversely, when $H < 1$, the time intervals tend to be larger, and thereby, its fractional FCDF coarser. For example, consider the exponential probability distribution with $0 < H < 1, F(t) = 1 - e^{-\mu t}$, then:

$$F(t) = -\sum_{k\geq 1} \frac{(-\mu t)^k}{k!}.$$

Since $D^H(t^a) = \frac{\Gamma(a+1)}{\Gamma(a+1-H)} t^{a-H}$, the fractional derivative of $F$ is

$$D^H F(t) = -\sum_{k\geq 1} \frac{(-\mu)^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1-H)} t^{k-H}$$

$$= -\left(\sum_{k\geq 1} \frac{(-\mu t)^k}{\Gamma(k+1-H)}\right) t^{-H}.$$

And therefore:

$$f_H(t) = -\left(\sum_{k\geq 1} \frac{(-\mu t)^k}{\Gamma(k+1-H)}\right) t^{-H}$$

and at $H = 1$,

$$f_1(t) = -\left(\sum_{k=1}^{\infty} \frac{(-\mu t)^k}{(k-1)!}\right) t^{-1} = \mu \sum_{k=0}^{\infty} \frac{(-\mu t)^k}{k!} = \mu e^{-\mu t}.$$

In other words, $F$ is an exponential CDF, and its derivative is indeed an exponential distribution. The fractional exponential probability distribution defined by $f_H(t)$ is not, however, a conventional distribution. Indeed, since $f_H = I^{1-H}(e^{-\mu t})$, it is easy to deduce that its integral over $[0, \infty[$ is infinite. Note that:

$$f_H(t) = \mu^{1-H} E_{1,2-H}(-\mu t)$$

where $E_{\alpha,\beta}(x)$ is the Mittag–Leffler function given by:

$$E_{\alpha,\beta}(x) = \sum_{n\geq 0} \frac{x^n}{\Gamma(n\alpha + \beta)}.$$

This raises a number of issues pertaining to the meaning of fractional probability distributions as alluded to previously. Recently, however, in a work-in-process by Simon, Boudabsa and Vallois, a complete fractional exponential distribution was devised. To do so, one may consider instead the fractional differential equation: $D^H F = \mu(1-F)\ \ 0 < H < 1$ and demonstrate that:

$$F(x) = \sum_{n\geq 0} \frac{(-\mu x^H)^n}{\Gamma(1+nH)} = E_{H,1}(-\mu x^H).$$

# 4. Example: Fractional Discrete Probability Distribution

Let $X$ be a discrete random variable with its CDF given by:

$$F(x) = \begin{cases} 0 & if \quad x < a_1 \\ \sum_{k=1}^{i} \alpha_k & if \quad a_i < x < a_{i+1} \\ \sum_{k=1}^{n} \alpha_k = 1 & if \quad x \geq a_n \end{cases}$$

Its fractional probability distribution defined and summed in the interval $a_i \leq x \leq a_{i+1}, i = 1,2,\dots,n,$ is thus:

$$f_H(x) = \frac{1}{\Gamma(1-H)}\left(\sum_{k=1}^{i} \frac{\alpha_k}{(x-a_k)^H}\right), a_i \leq x \leq a_{i+1}, i = 1,2,3,\dots,n.$$

The proof is as follows. In the interval $x \in (0, a_1], F(x) = 0,$ and, therefore $f_H(x) = 0$. In the interval $x \in [a_1, a_2)$, we note that:

$$I^{1-H}F(x) = \frac{1}{\Gamma(1-H)} \int_{a_1}^{x} (x-\tau)^{-H}\, \alpha_1 d\tau = \frac{\alpha_1 (x-a_1)^{1-H}}{\Gamma(2-H)},$$

$$a_1 < x < a_2.$$

Then,

$$f_H(x) = \frac{d}{dx} I^{1-H} f(x) = \frac{\alpha_1 (x - a_1)^{-H}}{\Gamma(1-H)}.$$

Repeating these calculations over each interval, $x \in [a_i, a_{i+1}), i \leq n - 1$, we obtain:

$$I^{1-H} F(x) = \frac{1}{\Gamma(1-H)} \sum_{k=1} \frac{\alpha_k}{(x - a_k)^H}$$ and recover $F_1 = F$ when $H = 1$.

The implications of a fractional discrete distribution are then: at each point $a_i$, the fractional probability $f_H$ has a singularity and a fat tail since:

$$f_H(x) \sim \frac{1}{\Gamma(1-H)} \frac{1}{x^H}, x \to \infty.$$

# 5. Fractional Distributions and Randomness

In practice, fitting data (that is, the real world) may modify the tools usually used in statistical modelling. For instance, the standard Brownian Motion can be replaced by a Lévy process so that new random variables have the "heavy tail distribution" property, $P(X_T > x) \sim \frac{C}{x^\alpha}$, where $x \to \infty$, for some $\alpha > 0$. In the same spirit, it could work with a fractional Brownian Motion with a Hurst index $H \in [0,1]$. In the following, we seek to introduce statistical fractional randomness and their associated probability distributions defined by the fractional Brownian Bridge for $1/2 < H \leq 1$ and $\alpha = 1/(1-H), 0 < H \leq 1/2$ for stable distributions. Although there is a huge body of literature on this topic, our intent here is limited to a statistical approach to fractional randomness.

**Definition 1:** Deterministic linear granularity

A granular system is a triplet $(O, X, (\varepsilon_t)_{t \in I})$, such that:

- $O$ is a fixed point on the real line;

- $X$ is the root, the origin of the granular structure;
- the size of the grain at level $t$ is $\varepsilon_t$ and induces a segmentation of the whole line via the collection of points $\{X + k\varepsilon_t, k \in \mathbb{Z}\}$;
- $t \mapsto \varepsilon_t$ is right-continuous decreasing (i.e., $0 < \varepsilon_s < \varepsilon_t$ for any $s < t$) and $\varepsilon_0 = 0$.

We say that this system is coherent if:

$$\varepsilon_{t+s} = \varepsilon_t + \varepsilon_s, t, s \geq 0. \tag{5.1}$$

The coherence property implies that the segmentation with root $X$ and granularity $\varepsilon_{t+s}$ is included in the thinner granular system with any root of the type $X + k\varepsilon_t$ and size of grain $\varepsilon_s$. Mathematically:

$$\{X + k(\varepsilon_t + \varepsilon_s), k \in \mathbb{Z}\} \subset \{X + k\varepsilon_s + k^!\varepsilon_t, k^!, k \in \mathbb{Z}\}.$$

When $I = [0,1]$, it is easy to deduce from 5.1 that $\varepsilon_t = ct$, for all $t > 0$, where $c > 0$ is a constant. We can give two examples. A microscope that has a finite number of possible zooms $(\varepsilon_k)_{1 \leq k \leq n}$, or markets, which are observed at different times: weekly, daily, hourly, per minute….

**Definition 2:** Stochastic Granularity

A stochastic granular system is a granular entity $\left(O, X, (\varepsilon_t)_{t \in [0,1]}\right)$ like above, such that: $X$ is a random variable and $(\varepsilon_t)_{t \in [0,1]}$ a collection of random variables, that is, a stochastic process that is independent from $X$ and:

$$\varepsilon_{t+s} = \varepsilon_t + \hat{\varepsilon}_s, \forall t, s \geq 0 \tag{5.2}$$

where $\hat{\varepsilon}_s$ is a random variable independent of $\varepsilon_t$ and has the same distribution as $\varepsilon_s$.

Let $(\varepsilon_t)_{t \geq 0}$ be a Lévy process, such that $\varepsilon_0 = 0$, independent and stationary increments. Then, the random variable $\varepsilon_{t+s} - \varepsilon_t$ is independent of $\varepsilon_t$ and is distributed as $\varepsilon_s$, and therefore (5.2) holds. The standard Brownian Motion $(B_t)_{t \geq 0}$ is a Lévy process, but is not valued in $[0, +\infty\,]$. Nevertheless, any non-increasing Lévy process (also called a subordinator) is convenient, and in particular, the stable ones.

# 6. Towards fractional integration

We begin with few preliminaries. Let $Y$ be a non-negative random variable with a Probability Density Function (PDF) $f\colon [0, \infty[ \to [0, +\infty[$ such that:

$$\int_0^\infty f(t)dt = 1.$$

Let $Y_1$ and $Y_2$ be two independent random variables with PDF $f_1$ and $f_2$, respectively, with the PDF of $Y_1 + Y_2$:

$$f(x) = f_1 * f_2(x) := \int_0^x f_1(t)f_2(x-t)dt, x > 0.$$

Returning to our granular system $\big(O, X, (\varepsilon_t)_{t\in[0,1]}\big)$ where $(\varepsilon_t)_{t\geq 0}$ is a subordinator, and assuming that $\varepsilon_t$ has a density function $f_t$ at any time $t > 0$, then:

$$f_t * f_s = f_{t+s}, t, s > 0. \tag{5.3}$$

Recalling that $X$ and $\varepsilon_t$ are independent and that $f$ is the PDF of $X$, then the PDF of $X + \varepsilon_t$ is:

$$\Lambda_t(x) := f * f_t(x) \equiv \int_0^x f(u)f_t(x-u)du, \qquad x > 0.$$

We have already introduced the function $\varphi_t$ in Section 3:

$$\varphi_t(x) := \frac{1}{\Gamma(t)} x^{t-1}\mathbf{1}_{x>0}, \qquad t > 0.$$

Then the family $(\varphi_t)_{t>0}$ is a semi-group, that is, it satisfies a property of the type (5.3):

$$\varphi_t * \varphi_s = \varphi_{t+s.}$$

$\varphi_t$, however, is not a PDF since:

$$\int_0^\infty \varphi_t(x)dx = +\infty.$$

This leads us to define for any $0 < H \leq 1$ the granular fractional density $F_H$ as:

$$F_H = f * \varphi_{H(x)} = I^H f(x) = \frac{1}{\Gamma(H)} \int_0^x f(t)(x-1)^{H-1}\, dt,\ x > 0 \quad (5.4)$$

where the Riemann–Liouville operator $I^H$ has been defined (Liouville, 1832). $F_H$ is formally the density of $X + \varepsilon_H^*$ where $\varepsilon_H^*$ is a pseudo-random variable with "density" $\varphi_H$. The family $(\varepsilon_t^*)_{t>0}$ satisfies some independence property of type (5.3).

It is convenient to introduce a (classical) non-negative random variable $X_f$ with density function $f$. Note that $F_1$ coincides with the Cumulative Distribution Function (CDF) of $X_f$. Using the rules of the fractional integral and derivatives operator, we get:

$$F_H = I^H f = I^H D I^1 f = I^H D F_1 = I^H D^H D^{1-H} F_1 = D^{1-H} F_1.$$

The relation $F_H = D^{1-H} F_1$ implies that $F_H$ is a fractional probability distribution with order $1 - H$ (Tapiero and Vallois, 2016). Since $F_H = I^H f$, we can also interpret $F_H$ as the fractional CDF, (cf. also the Definition in Section 3). Recall, as pointed out previously, $F_H$ is not a conventional CDF. It is clear that (5.4) implies that $F_H$ admits the following stochastic representation:

$$F_H(x) = \frac{1}{\Gamma(H)} \mathbf{E}\left(\left(x - X_f\right)_+^{H-1}\right),\ x \geq 0 \quad (5.5)$$

where $\left(x - X_f\right)_+^{H-1}$ equals $\left(x - X_f\right)^{H-1}$ if $X_f < x$ and $0$ otherwise. Furthermore, it is easy to check that the random variable $\left(x - X_f\right)_+^{H-1}$ is integrable. We can deduce from (5.5) a first approximation of $F_H(x)$.

Namely, let $X^1, \dots., X^n$ be a sample of $X_f$. Then, for any $t > 0$, the strong law of large numbers directly implies:

$$\frac{1}{n\,\Gamma(H)}\left((t - X^1)_+^{H-1} + \cdots + (t - X^n)_+^{H-1}\right)$$

converges a.s. to $F_H(t)$, as $n \to \infty$.

If $H = 1$, we recover the classical strong law of large numbers since $\left(x - X^i\right)_+^{H-1} = \mathbf{1}_{\{X^i < x\}}$. Furthermore, assume that $H > 1/2$. Under some additional assumptions, we can prove that the random variable $(x - X)_+^{H-1}$ is square integrable. This implies that we can apply the classical central limit theorem. For any $x \geq 0$, the sequence of random variables

$$\frac{1}{\sqrt{n}}\left(\frac{1}{\Gamma(H)}\sum_{i=1}^{n}\left(x - X^i\right)_+^{H-1} - nF_H(x)\right) \qquad (5.6)$$

converge in distribution as $n \to \infty$ to a Gaussian random variable $\Lambda_x$. We can determine the covariance of the random function $(\Lambda_x)_{x\geq 0}$ (see Tapiero and Vallois (2018a) on the fractional Brownian Bridge). We may also prove a representation of $(\Lambda_x)_{x\geq 0}$ in terms of a stochastic integral with respect to the Brownian Bridge $(BB(t), 0 \leq t \leq 1)$, which, for simplicity, we refer the reader to our paper (2018a). If, however, $f \coloneqq \mathbf{1}_{[0,1]}$ (i.e. $X_f$ is uniform in $[0,1]$), then for any $x$, $\Lambda_x$ is distributed as $BB^H(x)$, where $(BB^H(t), 0 \leq t \leq 1)$ is the fractional Brownian Bridge:

$$BB_H^U(t) = \frac{1}{\Gamma(H)}\int_0^t (t - u)^{H-1}\, dBB(u), \qquad 0 \leq t \leq 1.$$

Finally, when $0 < H < 1/2$, we have to change the normalization in (5.6), and the limit is a stable random variable with parameter $\alpha \coloneqq 1/(1 - H) \in ]1,2[$. Such a result can be found in Tapiero and Vallois (2018b).

# 7. Conclusion

Financial Data analysed by parametric mathematical and stochastic models are constructed and used to predict future events and financial prices as well as to invest and manage risks. These models, however, are hypotheses, which are justified by their purported rationality and statistical estimates. The statistical treatment and analysis of the sampled data alter the traditional approach to financial complete market pricing models. It consists of the expectation of future prices and an appropriate filtration and conditional martingale providing a unique price. These differences are embedded in their applications to discrete and "fractional time" defined by their calculus. For example, given a continuous-time stochastic process, discretized:

$$dx_t = f(t, x_t)dt + \sigma(t, x_t)dW(t)$$

and

$$x_{t+\Delta t} - x_t \simeq f(t, x_t)\Delta t + \sigma(t, x_t)(W_{t+\Delta t} - W_t)$$

where $\Delta t$ is small and $W(t)$ defines a Brownian Motion. Higher order approximations of $f(t, x_t)$ and $\sigma(t, x_t)$ may be used further using Milshtein approximations.

Our conclusion to this text is that randomness is also altered by its fractional application due to its fractional time. This *Prisme* has thereby shown that depending on the fractional index, fractional randomness based on the statistical approach to fractional calculus, as shown for a fractional index $1/2 < H < 1$, results in a Brownian Bridge, while for $0 < H < 1/2$, an alpha stable distribution results.

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**Charles Tapiero** is a Distinguished Professor of Financial Engineering and Technology Management and founder and Chair (2006–2016) of the Department of Finance and Risk Engineering at the Tandon School of Engineering at New York University. He was Assistant and Associate Professor at Columbia University and has also held academic positions at the Hebrew University, the University of Washington, Case Western Reserve University, ESSEC in France, the Bar Ilan University in Israel, where he founded the Financial Mathematics Programme, and Concordia University. Professor Tapiero has published over 17 Books and 450 academic papers. His fields of interest span financial engineering; fractional, multi-agent and global finance; computational and actuarial science; risk technology management; and supply-chain management.

**Pierre Vallois** is Professor Emeritus at the University of Lorraine. He was a researcher for the French National Centre for Scientific Research (CNRS) in the Laboratoire de Probabilités et Applications at the University Paris VI from 1984 to 1993 and a full Professor at the University of Lorraine from 1996 to 2018. Professor Vallois headed the Mathematics Department of the Faculty of Sciences of the University of Lorraine from 2009 to 2013 and the Charles Hermite Federation from 2013 to 2017. He has published over 100 papers in peer-reviewed journals. His current interests include stochastic processes, generalized stochastic calculus and modelling in biology, medicine and finance.

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